Journal of Statistical Physics, Vol. 36, Nos. 1/2, 1984

Borel Summability of the Perturbation Series in a Hierarchical $\lambda(\nabla\phi)^4$ Model

K. Gawędzki,[1,4,5] A. Kupiainen,[2] and B. Tirozzi[3]

Received January 25, 1984

We prove Borel summability of the perturbation series for the dielectric constant and the free energy density for the hierarchical $\lambda(\nabla\phi)^4$ lattice model. Our methods are based on nonperturbative renormalization group analysis of the model.

KEY WORDS: Renormalization group; Borel summability; hierarchical model.

1. INTRODUCTION

The question what is the exact relation of the perturbation series to the nonperturbatively defined quantities in models of statistical mechanics and quantum field theory is fundamental, especially in the latter case, where the existence of the nonperturbative quantities is still to be shown in four space-time dimensions. After the early phenomenological and theoretical success of the renormalized perturbation calculus, some doubts arose whether the perturbation series determines the whole structure as it was found[1,2] that it diverges in many interesting cases. A possible way out of this difficulty seemed to be offered by the idea of nonconventional resummation of divergent series, the Borel resummation technique being the best-known example of such a procedure.[3] Since then, it has been shown

[1] C.N.R.S., I.H.E.S., 91440 Bures-sur-Yvette, France.
[2] Helsinki University of Technology, Department of Technical Physics, ESP0015, Finland.
[3] Istituto di Matematica, Università di Roma, 00185 Roma, Italy.
[4] On leave from the Department of Mathematical Methods of Physics, Warsaw University, Poland.
[5] Supported in part by the Center for Interdisciplinary Research, Bielefeld University, Germany.

0022-4715/84/0700-0145$03.50/0

that the perturbation series of super-renormalizable models of field theory (like $\lambda\phi^4$ in two and three dimensions) is Borel summable.[4,5] The study of these models contains elements of local analysis in the phase space which can be best systematized by the renormalization group approach.[6,7] In the present paper we begin the study of the infrared counterpart of the field theoretic (ultraviolet) problem: the Borel summability of the perturbation theory for lattice critical models. The simplest of such models, which is asymptotically free in infrared and in a certain sense infrared super-renormalizable, and for which we cannot hope for convergence of the perturbation series, is the $\lambda(\nabla\phi)^4$ model. This model has been studied in numerous works.[8–13] The analysis of[14,11,13] was based on the renormalization group ideas. Here, we show how these ideas may be used to prove the Borel summability of the perturbation series in such a case. For simplicity, we shall study a hierarchical model for which the renormalization group tranformation becomes a recursion for the single spin potential. We plan to deal with the full $\lambda(\nabla\phi)^4$ model in another publication. Being greatly simpler, the hierarchical model offers nevertheless an understanding which is sufficient for the general case. It might also provide a new insight into the summability problem of the infrared asymptotically free $\lambda(\phi^4)$ in four dimensions[14,15] and into the (infrared) renormalons.[16]

2. DEFINITIONS AND RESULTS

The $\lambda(\nabla\varphi)^4$ lattice model of equilibrium statistical mechanics, also called the anharmonic crystal, is described by the Gibbs state formally given as

$$\frac{1}{N}\exp\left[-\sum_{\mu,x}\lambda(\nabla_\mu\varphi_x)^4-\frac{1}{2}\sum_{\mu,x}(\nabla_\mu\varphi_x)^2\right]\prod_x d\varphi_x \tag{2.1}$$

Restricting ourselves to expectations expressible in terms of $\nabla_\mu\varphi\equiv\Phi_\mu$ (which is necessary in $d=2$), we may rewrite (2.1) as

$$\frac{1}{N}\exp\left(-\sum_{\mu,x}\lambda\Phi_{\mu x}^4\right)d\mu_G(\Phi) \tag{2.2}$$

where $d\mu_G(\Phi)$ stands for the Gaussian measure with mean zero and covariance

$$G_{xy}=\nabla(-\Delta)^{-1}\nabla(x-y) \tag{2.3}$$

Our hierarchical approximation,[15] similar to the one introduced in Ref. 17, consists of replacing the covariance G_{xy} by a hierarchical one G^h_{xy} described below, which mimics the long distance behavior of $G_{xy}\sim O(|x-y|^{-d})$. For simplicity, we shall choose G^h_{xy} to be diagonal in the vector indices of Φ and shall study only a single component model with the field $\Phi_{\mu_0} \equiv \Phi$.

Before we define G^h_{xy} we need some notations. Let L be an even integer and let for $x \in \mathbb{Z}^d$, x_k denote the point in $\mathbb{Z}^d$ whose coordinates are the integral parts of the coordinates of $L^{-k}x$. In other words $L^k x_k$ is the center of the $L^k \times \cdots \times L^k$ block to which x belongs. Choose a function A on the $L \times \cdots \times L$ block in $\mathbb{Z}^d$ centered at the origin which takes values ± 1 and sums up to zero. Define first

$$\Gamma_{xy} = A(x - Lx_1)A(y - Ly_1)\delta_{x_1 y_1} \tag{2.4}$$

Γ is a positive (but not strictly positive) matrix. The hierarchical covariance G^h (for a single field component) will be given as a superposition of a hierarchy of Γ's taken on different length scales:

$$G^h_{xy} = \sum_{k=0}^{\infty} L^{-dk}\Gamma_{x_k y_k} \tag{2.5}$$

Notice that

$$G^h_{xy} = O(L^{-dk_0}) \sim O(|x - y|^{-d}) \tag{2.6}$$

where k_0 is the first integer such that $x_{k_0} = y_{k_0}$.

In general, given a (reasonable) single spin potential v, we shall define the perturbed state [mimicking (2.2)] by

$$\frac{1}{N}\exp\left[-\sum_{x \in \Lambda} v(\Phi_x)\right] d\mu_{G^h}(\Phi) \tag{2.7}$$

For simplicity, we shall take the finite volume region Λ as $\{x \in \mathbb{Z}^d : x_{L^{N_0}} = 0\}$, i.e., as an $L^{N_0} \times \cdots \times L^{N_0}$ block centered at the origin.

(2.7) is specially well suited for the renormalization group analysis. Notice that by (2.5)

$$G^h_{xy} = L^{-d}G^h_{x_1 y_1} + \Gamma_{xy} \tag{2.8}$$

and by (2.4),

$$\sum_{x\,:\,x_1 \text{ fixed}} \Gamma_{xy} = 0 \tag{2.9}$$

Decomposing

$$\Phi_x = L^{-d/2}\Phi^1_{x_1} + Z_x \tag{2.10}$$

where the block spin field

$$\Phi^1_{x_1} = L^{-d/2} \sum_{x\,:\,x_1 \text{ fixed}} \Phi_x \tag{2.11}$$

we easily see from (2.8) and (2.9) that if Φ is distributed with the measure

$d\mu_{G^h}$ then (1) so is Φ^1, (2) Φ^1 and Z are independent so that

$$d\mu_{G^h}(\Phi) = d\mu_{G^h}(\Phi^1)\, d\mu_\Gamma(Z) \tag{2.12}$$

For the perturbed state (2.7), we get the following expression:

$$\frac{1}{N}\exp\Big[-\sum_{x\in\Lambda} v\big(L^{-d/2}\Phi^1_{x_1} + Z_x\big)\Big]\, d\mu_{G^h}(\Phi^1)\, d\mu_\Gamma(Z) \tag{2.13}$$

From (2.13) we easily read off the effective state for the block spin field Φ^1. It is given by

$$\frac{1}{N}\int \exp\Big[-\sum_{x\in\Lambda} v\big(L^{-d/2}\Phi^1_{x_1} + Z_x\big)\Big]\, d\mu_\Gamma(Z)\, d\mu_{G^h}(\Phi^1) \tag{2.14}$$

The crucial simplification of the hierarchical model consists in making the fluctuation measure $d\mu_\Gamma(Z)$ local over $L\times\cdots\times L$ blocks (in the case of the full model we would have coupling of different blocks by exponentially decaying tail terms). Thus (2.14) may be written in the same form as (2.7):

$$\frac{1}{N}\exp\Big[-\sum_{x_1\in\Lambda_1} v_1\big(\Phi^1_{x_1}\big)\Big]\, d\mu_{G^h}(\Phi^1) \tag{2.15}$$

where $\Lambda_1 = \{y\in\mathbb{Z}^d : y_{L^{N_0-1}} = 0\}$ and

$$\exp\big[-v_1\big(\Phi^1_{x_1}\big)\big] = \text{const}\int\exp\Big[-\sum_{x\,:\,x_1\text{ fixed}} v\big(L^{-d/2}\Phi^1_{x_1} + Z_x\big)\Big]\, d\mu_\Gamma(Z) \tag{2.16}$$

Using the specific form (2.4) of Γ and choosing the constant in (2.16) so that $v_1(0) = 0$, we may rewrite it as

$$\exp\big[-v_1(\Phi)\big] = \frac{\int\exp\big[-(1/2)L^d\sum_\pm v(L^{-d/2}\Phi\pm z) - (1/2)z^2\big]\big[dz/(2\pi)^{1/2}\big]}{\int\exp\big[-(1/2)L^d\sum_\pm v(\pm z) - (1/2)z^2\big]\big[dz/(2\pi)^{1/2}\big]} \tag{2.17}$$

Hence the renormalization group approach reduces the study of the massless state (2.7) to the analysis of iterations of the recursion (2.17) for single spin potential. For the partition function

$$Z_\Lambda(v) = \int\exp\Big[-\sum_{x\in\Lambda} v(\Phi_x)\Big]\, d\mu_{G^h}(\Phi) \tag{2.18}$$

the same approach gives immediately the recursion

$$Z_\Lambda(v) = \left\{\int\exp\Big[-\frac{1}{2}L^d\sum_\pm v(\pm z) - \frac{1}{2}z^2\Big]\frac{dz}{(2\pi)^{1/2}}\right\}^{|\Lambda_1|} Z_{\Lambda_1}(v_1) \tag{2.19}$$

In Ref. 15 we have shown that for a large class of small even v's the subsequent iterations v_n converge to $v_\infty(\phi) = (1/2)L^{-d}(\epsilon_\infty - 1)\phi^2$. ϵ_∞ has an interpretation of the dielectric constant as on long distances the infinite-volume two-point function $\langle \Phi_x \Phi_y \rangle$ becomes $\epsilon_\infty^{-1} G_{xy}^h$; see Ref. 15. In the same case the free energy density f_∞ is given by

$$f_\infty = - \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \log Z_\Lambda(v)$$

$$= - \sum_{k=0}^{\infty} L^{-d(k+1)}$$

$$\times \log \int \exp\left[-L^d v_k(z) - \frac{1}{2} z^2 \right] \frac{dz}{(2\pi)^{1/2}} \tag{2.20}$$

as easily follows from (2.19).

In the present paper, we shall consider the initial potential v depending on the parameter λ [e.g., $v_\lambda(\phi) = \lambda\phi^4$] and study the Borel summability of the perturbation expansion in powers of λ for the free energy density $f_\infty(\lambda)$ and the dielectric constant $\epsilon_\infty(\lambda)$. The results may be easily generalized to the infinite-volume correlation functions.

We shall make use of the Nevanlinna–Sokal (N–S) theorem, which establishes conditions under which a complex function $f(\lambda)$ is equal to the Borel sum $g(\lambda)$ of its asymptotic Taylor series.[18] So let $\sum_{n=0}^{\infty} a_n \lambda^n$ be a formal power series. We say that it is Borel summable in a domain C, if

(a) $B(t) = \sum_n a_n t^n / u!$ converges in some circle $|t| < \delta$.

(b) $B(t)$ has an analytic continuation to a neighborhood of the positive real axis.

(c) For $\lambda \in C$, $g(\lambda) = (1/\lambda) \int_0^\infty e^{-t/\lambda} B(t)\, dt$ converges (not necessarily absolutely).

$B(t)$ is called the Borel transform of the series $\sum_{n=0}^{\infty} a_n \lambda^n$ and $g(\lambda)$ is its Borel sum.

Now suppose that $f(\lambda)$ is analytic in

$$C_R \equiv \left\{ \lambda : \mathrm{Re} \frac{1}{\lambda} > \frac{1}{R} \right\} \tag{2.21}$$

$R > 0$ (see Fig. 1) and that

$$f(\lambda) = \sum_{n=0}^{N-1} a_n \lambda^n + R_N(\lambda) \tag{2.22}$$

with

$$|R_N(\lambda)| \leqslant A\sigma^N N!\, |\lambda|^N \tag{2.23}$$

uniformly in N and in $\lambda \in C_R$.

The N–S theorem asserts that the series $\sum_{n=0}^{\infty} a_n \lambda^n$ is Borel summable in C_R and that $f(\lambda) = g(\lambda)$ there. Applying this theorem, we shall obtain the following.

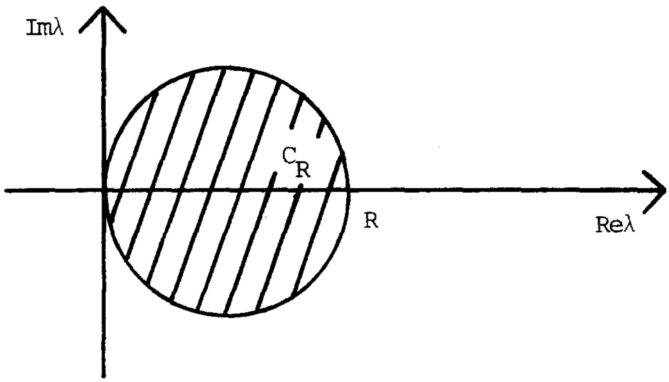

Fig. 1.

The Main Result: Suppose that L is big enough. If $v(\phi) = \lambda\phi^4$ then there exists $R > 0$ such that the infinite-volume dielectric constant $\epsilon_\infty(\lambda)$ and the free energy density $f_\infty(\lambda)$ are analytic in C_R and equal there to the Borel sums of their perturbative expansions.

3. INDUCTIVE RENORMALIZATION GROUP ANALYSIS

Our strategy is to establish the convergence of the iterates v_n of transformation (2.17) to a fixed point (this has been done in Ref. 15) and to prove the estimates of the type of (2.23) for each v_n. This requires some additional work.

Let us consider the Boltzmann factors depending on $\lambda \in C_R$ for some $R > 0$:

$$g_n(\phi) = e^{-v_n(\phi)} \tag{3.1}$$

with

$$v_n(\phi) = \tfrac{1}{2} L^{-d}(\epsilon_n - 1)\phi^2 + \tilde{v}_n(\phi) \tag{3.2}$$

$$\tilde{v}_n(0) = \frac{\partial^2}{\partial\phi^2}\tilde{v}_n(0) = 0 \tag{3.3}$$

$$\tilde{g}_n(\phi) = g_n(\phi)\exp\left[\tfrac{1}{2} L^{-d}(\epsilon_n - 1)\phi^2\right] \tag{3.4}$$

We assume inductively that

(A_n): $\tilde{g}_n(\phi)$ is analytic in ϕ in the strip $|\mathrm{Im}\,\phi| < (n_0 + n)^2$ and in λ in C_R and satisfies there the estimate

$$\left|\frac{\partial^m}{\partial\lambda^m}\tilde{g}_n(\phi)\right| \leqslant C_n^m (m!)^2 e^{\kappa_n|\phi|^2}, \qquad m = 0, 1, \ldots \tag{3.5}$$

(B_n): For $|\phi| < (n_0 + n)^2$, $g_n(\phi) = e^{-v_n(\phi)}$ for v_n analytic in ϕ and in $\lambda \in C_R$. ϵ_n and $\tilde{v}_n$ as given by (2) and (3) satisfy

$$\left| \frac{d^m}{d\lambda^m} (\epsilon_n^{-1} - 1) \right| \leqslant E_n C_n^m (m!)^2 \tag{3.6}$$

$$\left| \frac{\partial^m}{\partial\lambda^m} \tilde{v}_n(\phi) \right| \leqslant \eta_n C_n^m (m!)^2 \tag{3.7}$$

Proposition 1. Fix some $0 < \delta < 1$ and set $\eta_n = \delta^{n_0+n}$, $\kappa_n = \kappa_0 \prod_{k=1}^n [1 + (n_0 + k)^{-3/2}]$, $C_n = C_0 \prod_{k=1}^n [1 + (n_0 + k)^{-3/2}]$, $E_n = \delta^{n_0/2} \prod_{k=1}^n [1 + (n_0 + k)^{-3/2}]$. Suppose that $L \geqslant \bar{L}(\delta)$, $0 < \kappa_0 \leqslant \bar{\kappa}_0(\delta, L)$, $n_0 \geqslant \bar{n}_0(\delta, L, \kappa_0)$, $C_0 \geqslant \bar{C}_0(\delta, L, \kappa_0, n_0)$, and $R \leqslant \bar{R}(\delta, L, \kappa_0, n_0, C_0)$. Then (A_n), (B_n) for g_n implies (A_{n+1}), (B_{n+1}) for g_{n+1}.

Notice that if one considers the conditions (A_n), (B_n) for $m = 0$ only then Proposition 1 essentially coincides with the result of Ref. 15, which shows the convergence of v_n's to a Gaussian fixed point.

In the proof of Proposition 1, we shall keep using the following simple result:

Lemma 1. Suppose that h is an analytic function of λ in some domain and that

$$\left| \frac{d^m}{d\lambda^m} h \right| \leqslant A (m!)^2 C^m, \qquad C, A > 0, \quad m = 0, 1, \ldots \tag{3.8}$$

Then for any integer $k > 0$,

$$\left| \frac{d^m}{d\lambda^m} h^k \right| \leqslant A^k C^m (m!)^2 \binom{m + k - 1}{k - 1} \tag{3.9}$$

and for any function $f(u)$ analytic for $|u| < R_f$, $R_f > A$, bounded there by C_f and vanishing at zero,

$$\left| \frac{d^m}{d\lambda^m} f \circ h \right| \leqslant \frac{C_f}{1 - A R_f^{-1}} A R_f^{-1} \left(\frac{C}{1 - A R_f^{-1}} \right)^m (m!)^2 \tag{3.10}$$

Proof of Lemma 1.

$$\left| \frac{d^m}{d\lambda^m} h^k \right| \leqslant \sum_{\substack{(m_1, \ldots, m_k) \\ m_i \geqslant 0, \sum_i m_i = m}} \frac{m!}{\prod_i m_i!} \prod_{i=1}^k \left[A C^{m_i} (m_i!)^2 \right] \tag{3.11}$$

$$\leqslant A^k C^m (m!)^2 \sum_{\substack{(m_1, \ldots, m_k) \\ m_i \geqslant 0, \sum_i m_i = m}} 1 = A^k C^m (m!)^2 \binom{m + k - 1}{k - 1} \tag{3.11}$$

Using (3.11) and the Cauchy estimates for the Taylor coefficients of f, we obtain

$$\left|\frac{d^m}{d\lambda^m} f\circ h\right| \leqslant C_f \sum_{k=1}^{\infty} R_f^{-k}\left|\frac{d^m}{d\lambda^m} h^k\right|$$
$$\leqslant C_f C^m (m!)^2 \sum_{k=1}^{\infty} \frac{(m+k-1)!}{m!(k-1)!}\left(AR_f^{-1}\right)^k \qquad (3.12)$$

Setting $\xi \equiv AR_f^{-1}$ gives

$$\left|\frac{d^m}{d\lambda^m} f\circ h\right| \leqslant C_f C^m m!\,\xi \sum_{k=1}^{\infty} \frac{d^m}{d\xi^m}\xi^{m+k-1}$$
$$= C_f C^m m!\,\xi \frac{d^m}{d\xi^m}\frac{\xi^m}{1-\xi} = C_f C^m m!\,\xi \sum_{p=0}^{m}\binom{m}{p}\frac{\xi^p m!}{(1-\xi)^{p+1}}$$
$$= C_f \frac{\xi}{1-\xi}\left(\frac{C}{1-\xi}\right)^m (m!)^2 \quad \blacksquare \qquad (3.13)$$

The typical situation in which we shall apply Lemma 1 is when A is very small and R_f and C_f are $O(1)$. As an example take $h=\epsilon_n^{-1}-1$ and $f(u)=-u/(u+1)$ $R_f=1/2$, $C_f=1$. Then Lemma 1 together with (3.6) give

$$\left|\frac{d^m}{d\lambda^m}(\epsilon_n-1)\right| \leqslant \frac{1}{1-2E_n}2E_n\left(\frac{C_n}{1-2E_n}\right)^m (m!)^2$$
$$\leqslant 3E_n(2C_n)^m(m!)^2 \qquad (3.14)$$

Let $z_i=L^{-d/2}\phi+z$ for $i=1,\ldots,L^d/2$ and $z_i=L^{-d/2}\phi-z$ for $i=L^d/2+1,\ldots,L^d$. Denote by $d\mu_\gamma(z)$ the Gaussian measure $\exp[-(1/2)\gamma^{-1}z^2]dz/(2\pi\gamma)^{1/2}$. Define

$$g'_{n+1}(\phi)=\int\prod_{i=1}^{L^d}\tilde{g}_n(z_i)\,d\mu_{\epsilon_n^{-1}}(z)\Big/\int\prod_{i=1}^{L^d}\tilde{g}_n(z)\,d\mu_{\epsilon^{-1}}(z) \qquad (3.15)$$

where $\tilde{g}_n$ is related to the Boltzmann factor g_n by (3.4). Comparison with the renormalization group recursion (2.17) gives

$$g_{n+1}(\phi)=\exp\left[-(1/2)L^{-d}(\epsilon_n-1)\phi^2\right]g'_{n+1}(\phi) \qquad (3.16)$$

In order to estimate the mth derivative with respect to λ for $m>0$, we shall make use of the following integration-by-parts identity:

$$\frac{d}{d\lambda}\int F(z)\,d\mu_{\epsilon_n^{-1}}(z)=\frac{1}{2}\left(\frac{d}{d\lambda}\epsilon_n^{-1}\right)\int\frac{d^2F(z)}{dz^2}\,d\mu_{\epsilon_n^{-1}}(z) \qquad (3.17)$$

Applying (3.17) to the numerator of the right-hand side of (3.15), which we denote by $h(\phi)$, one obtains

$$\frac{\partial^m}{\partial\lambda^m} h(\phi) = \sum_{\substack{(I_i)_{i=1}^{L^d},\ |I_i| \geqslant 0 \\ \{I_\alpha\}_{\alpha=1}^{l},\ |I_\alpha| > 0}} \int \prod_\alpha \left(\frac{1}{2} \frac{\partial^{|I_\alpha|}}{\partial\lambda^{|I_\alpha|}} \epsilon_n^{-1} \right) \frac{\partial^{2l}}{\partial z^{2l}} \left[\prod_i \frac{\partial^{|I_i|}}{\partial\lambda^{|I_i|}} \tilde{g}_n(z_i) \right] d\mu_{\epsilon_n^{-1}}(z) \tag{3.18}$$

with $\{I_i\}, \{I_\alpha\}$ disjoint, $(\bigcup_i I_i) \cup (\bigcup_\alpha I_\alpha) = \{1, \ldots, m\}$ and we should consider the collection (I_i) as ordered and $\{I_\alpha\}$ as unordered.

We shall analyze (3.18) separately for small and for large values of ϕ.

Small Fields

Insert into (3.18) the partition of unity $1 = \chi(z) + \chi^\perp(z)$, where χ is the characteristic function of the set $\{z : |z| < \epsilon(n_0 + n)^2\}$ for some small fixed $\epsilon > 0$. Let us take $|\phi| < L^{d/2}(1 - 2\epsilon)(n_0 + n + 1)^2$. With the use of the bound (3.7) and Lemma 1 with $f(u) = e^u - 1$, $R_f = 1$, we obtain the following estimate for z in the support of χ, $|\zeta| < n_0 + n$, and $|I_i| > 0$ ($|L^{-d/2}\phi \pm z_i + \zeta| < (n_0 + n)^2$):

$$\left| \frac{\partial^{|I_i|}}{\partial\lambda^{|I_i|}} \tilde{g}_n(z_i + \zeta) \right| \leqslant \frac{(e-1)\eta_n}{1 - \eta_n} \left(\frac{C_n}{1 - \eta_n} \right)^m (m!)^2$$
$$\leqslant 2\eta_n C_n'^m (m!)^2 \tag{3.19}$$

where we shall denote by C_n' different constants increasing in the process of estimation but always bounded by $C_n(1 + (n_0 + n + 1)^{-3/2}) = C_{n+1}$. For $|I_i| = 0$, we have

$$|\tilde{g}_n(z_i + \zeta)| \leqslant e^{\eta_n} \tag{3.20}$$

Thus

$$\left| \prod_{i=1}^{L^d} \frac{\partial^{|I_i|}}{\partial\lambda^{|I_i|}} \tilde{g}_n(z_i + \zeta) \right| \leqslant \prod_{i : |I_i| > 0} \left(2\eta_n C_n'^{|I_i|} (|I_i|!)^2 \prod_{i : |I_i| = 0} e^{\eta_n} \right) \tag{3.21}$$

Using the Cauchy formula to estimate $\partial^{2l}/\partial z^{2l}$, (3.21) and (3.6), we may bound the integrand P on the right-hand side of (3.18) for z in the support of z:

$$|P| \leqslant \left[\prod_\alpha \frac{1}{2} E_n C_n^{|I_\alpha|} (|I_\alpha|!)^2 \right] \left[\prod_{i : |I_i| > 0} 2\eta_n C_n'^{|I_i|} (|I_i|!)^2 \right]$$
$$\times \left(\prod_{i : |I_i| = 0} e^{\eta_n} \right) (2l)! \, (n_0 + n)^{-2l} \tag{3.22}$$

If all $|I_i| = 0$, we may obtain additional factor $L^d\eta_n$ using $|\prod_i \tilde{g}_n(z_i + \zeta) - 1| \leqslant L^d\eta_n \prod_i e^{\eta_n}$ (remember that we consider $m > 0$ case).

Let us insert (3.22) to the part $h'_m(\phi)$ of (3.18) obtained by restricting the z integration to the support of $\chi(z)$. Let us also order the collection $\{I_\alpha\}$ and resum all $(I_i), (I_\alpha)$ with given $|I_i| = m_i$ and $|I_\alpha| = m_\alpha$. We obtain

$$|h'_m(\phi)| \leqslant C_n'^m \sum_{\substack{(m_i),(m_\alpha)\\ m_i \geqslant 0,\, m_\alpha > 0\\ \sum_i m_i + \sum_\alpha m_\alpha = m}} \frac{m!}{\prod_i m_i! \prod_\alpha m_\alpha!} \frac{2^{-l} E_n^l (2l)!}{l!\,(n_0 + n)^{2l}}$$
$$\times \left(\prod_{i\,:\,m_i > 0} 2\eta_n (m_i!)^2 \right) \left(\prod_{i\,:\,m_i = 0} e^{\eta_n} \right) \prod_\alpha (m_\alpha!)^2 (L^d \eta_n) \tag{3.23}$$

where the last factor $L^d\eta_n$ appears only in the terms with all $m_i = 0$.

We shall estimate first the sum over terms with at least one $m_i > 0$. Call it the 1st part. To fight with the additional $2l!/l! \leqslant 4^l l!$ we shall use the following easy estimate:

$$\frac{\left(\prod_{\alpha=1}^{l} m_\alpha! \right) l!}{\left(\sum_{\alpha=1}^{l} m_\alpha \right)!} \leqslant 1 \tag{3.24}$$

Thus

$$\text{1st part} \leqslant C_n'^m (m!)^2 \sum_{j=1}^{L^d} \binom{L^d}{j} (2\eta_n)^j (e^{\eta_n})^{L^d - j} \sum_{\substack{(m_i)_{i=1}^j,\, m_i > 0\\ (m_\alpha)_{\alpha=1}^l,\, m_\alpha > 0\\ \sum m_i + \sum m_\alpha = m}} \left[\frac{2E_n}{(n_0 + n)^2} \right]^l$$
$$\leqslant 3\eta_n C_n'^m (m!)^2 \sum_{j=1}^{L^d} \binom{L^d}{j} (2\eta_n)^{[(j-1)/2]\cdot 2}$$
$$\times \sum_{l=0}^{m-j} \binom{m-1}{j+l-1} \left(\frac{2E_n}{(n_0 + n)^2} \right)^l$$
$$\leqslant 3\eta_n C_n'^m (m!)^2 \sum_{j=1}^{L^d} \binom{L^d}{j} (2\eta_n)^{(j-1)/2}$$
$$\times \sum_{l=0}^{m-j} \binom{m-1}{j+l-1} \left(\frac{2E_n}{(n_0 + n)^2} \right)^{l+j-1} \tag{3.25}$$

where we have used

$$(e^{\eta_n})^{L^d-j} \leqslant \frac{3}{2}, \qquad (2\eta_n)^{1/2} \leqslant \frac{2E_n}{(n_0+n)^2} \tag{3.26}$$

Now

$$\begin{aligned}\sum_{l=0}^{m-j}\binom{m-1}{j+l-1}\left(\frac{2E_n}{(n_0+n)^2}\right)^{l+j-1} &\leqslant \sum_{l=0}^{m-1}\binom{m-1}{l}\left(\frac{2E_n}{(n_0+n)^2}\right)^{l} \\ &= \left[1+\frac{2E_n}{(n_0+n)^2}\right]^{m-1}\end{aligned} \tag{3.27}$$

So

$$\begin{aligned}\text{1st part} &\leqslant 3\eta_n C_n'^m (m!)^2 \sum_{j=1}^{L^d}\binom{L^d}{j}(2\eta_n)^{(j-1)/2}\left[1+\frac{2E_n}{(n_0+n)^2}\right]^{m-1} \\ &\leqslant 3L^d\eta_n C_n'^m (m!)^2\left[1+O(\eta_n^{1/2})\right]\left[1+\frac{4E_n}{(n_0+n)^2}\right]^{m-1} \\ &\leqslant 3L^d\eta_n C_n'^m (m!)^2\end{aligned} \tag{3.28}$$

where of course in the last step C_n' increased slightly again.

For the 2nd part of the right-hand side of (3.23) containing terms with all $m_i = 0$, we obtain

$$\begin{aligned}|\text{2nd part}| &\leqslant L^d\eta_n e^{L^d\eta_n} C_n'^m (m!)^2 \sum_{l=1}^{m}\binom{m-1}{l-1}\left[\frac{2E_n}{(n_0+n)^2}\right]^l \\ &\leqslant \frac{2E_n}{(n_0+n)^2} L^d\eta_n C_n'^m (m!)^2 \sum_{l=0}^{m-1}\binom{m-1}{l}\left(\frac{2E_n}{(n_0+n)^2}\right)^l \\ &\leqslant \frac{2E_n}{(n_0+n)^2} L^d\eta_n C_n'^m (m!)^2\end{aligned} \tag{3.29}$$

Putting (3.28) and (3.29) together, we obtain

$$|h_m'(\phi)| \leqslant 3L^d\eta_n C_n'^m (m!)^2 \tag{3.30}$$

Next, we shall estimate the contribution $h''(\phi)$ to (3.18) from the integration over the support of $\chi^\perp(z)$. All the time for $|\phi| < L^{d/2}(1-2\epsilon)(n_0+n+1)^2$ and $|\zeta| < n_0+n$ but for $|z| \geqslant \epsilon(n_0+n)^2$

$$\left|\frac{\partial^{|I_i|}}{\partial\lambda^{|I_i|}}\, \tilde{g}_n(z_i+\zeta)\right| \leqslant C_n^{|I_i|}(|I_i|!)^2 e^{\kappa_n|z_i+\zeta|^2} \tag{3.31}$$

where we have used (3.5). Using again the Cauchy formula, we found the integrand P on the right-hand side of (3.18):

$$|P| \leqslant \left[\prod_{\alpha} \frac{1}{2} E_n C_n^{|I_\alpha|} (|I_\alpha|!)^2 \right] \left[\prod_i C_n^{|I_i|} (|I_i|!)^2 \right]$$
$$\times \exp\left(\kappa_n |\phi|^2 + 2L^d \kappa_n z^2 + 2L^d \kappa_n |\zeta|^2 \right) \tag{3.32}$$

Now

$$\exp\left(\kappa_n|\phi|^2\right) \int \chi^{\perp}(z) \exp\left(2L^d\kappa_n z^2\right) d\mu_{\epsilon_n^{-1}}(z)$$
$$= \exp\left(\kappa_n|\phi|^2\right) \int \chi^{\perp}(z) \exp\left[\frac{1}{2}(\epsilon_n - 4L^d\kappa_n) z^2 \right] \frac{dz}{\left(2\pi\epsilon_n^{-1}\right)^{1/2}}$$
$$\leqslant \exp\left[\kappa_n L^d (1-2\epsilon)^2 (n_0+n+1)^4 - \frac{1}{4}\epsilon^2 (n_0+n)^4 \right]$$
$$\leqslant \exp\left[-\frac{1}{8}\epsilon^2 (n_0+n)^4 \right] \tag{3.33}$$

where in the last step we have used the smallness of κ_n. Hence we may estimate $|h''(\phi)|$ by, e.g., the right-hand side of (3.23) multiplied by $\exp[-(1/10)\epsilon^2(n_0+n)^4]$. As a result,

$$|h_m''(\phi)| \leqslant 3L^d \eta_n C_n'^m (m!)^2 \exp\left[-\frac{1}{10}\epsilon^2 (n_0+n)^4 \right] \tag{3.34}$$

Putting (3.30) and (3.35) together, we obtain

$$\left| \frac{\partial^m h(\phi)}{\partial \lambda^m} \right| \leqslant 3L^d \eta_n C_n'^m (m!)^2 \tag{3.35}$$

It is easy to see that (3.35) holds also for $m = 0$ if we replace $h(\phi)$ by $h(\phi) - 1$ (in fact, this case was already considered in Ref. 15). Thus

$$\left| \frac{\partial^m}{\partial \lambda^m} \left[\int \prod_{i=1}^{L^d} \tilde{g}_n(z_i)\, d\mu_{\epsilon_n^{-1}}(z) - 1 \right] \right| \leqslant 3L^d \eta_n C_n'^m (m!)^2 \tag{3.36}$$

Once more we shall apply Lemma 1, this time for $f(u) = \log(1+u)$, $R_f = 1/2$, and obtain

$$\left| \frac{\partial^m}{\partial \lambda^m} \log \int \prod_{i=1}^{L^d} \tilde{g}_n(z_i)\, d\mu_{\epsilon_n^{-1}}(z) \right| \leqslant \frac{\log 2}{1 - 6L^d\eta_n} 6L^d \eta_n \left(\frac{C_n'}{1 - 6L^d\eta_n} \right)^m (m!)^2$$
$$\leqslant 6L^d \eta_n C_n'^m (m!)^2 \tag{3.37}$$

By the very definition of $\tilde{v}_{n+1}$, we have

$$\begin{aligned}\tilde{v}_{n+1}(\phi) = &-\log\int\prod_i \tilde{g}_n(z_i)\, d\mu_{\epsilon_n^{-1}}(z)\\ &+\log\int\prod_i \tilde{g}_n(z)\, d\mu_{\epsilon_n^{-1}}(z)\\ &+\frac{1}{2}\frac{\partial^2}{\partial\phi^2}\restriction_{\phi=0}\left[\log\int\prod_i \tilde{g}_n(z_i)\, d\mu_{\epsilon_n^{-1}}(z)\right]\phi^2 \end{aligned}\tag{3.38}$$

In order to obtain the final bound for $(\partial^m/\partial\lambda^m)\tilde{v}_{n+1}(\phi)$, we apply the contraction lemma used many times in Ref. 15.

Lemma 2. Let $v(\phi)$ be analytic for $|\phi| < R$ and let $|v(\phi)|$ be bounded in the same region by the constant C. Then for $|\phi| < r < R$

$$\left|v(\phi) - \sum_{k=0}^{n}\frac{1}{k!}\phi^k\frac{d^k}{d\phi^k}v(0)\right| \leqslant \frac{1}{1-r/R}\left(\frac{r}{R}\right)^{n+1}C \tag{3.39}$$

Indeed, define

$$w(\phi) = v(\phi) - \sum_{k=0}^{n}\frac{1}{k!}\phi^k\frac{d^k}{d\phi^k}v(0) \tag{3.40}$$

Then by the Cauchy formula,

$$w(\phi) = \frac{1}{2\pi i}\int_{|\zeta|=R-\epsilon} v(\zeta)\left(\frac{\phi}{\zeta}\right)^{n+1}\frac{1}{\zeta-\phi}\,d\zeta \tag{3.41}$$

and (3.39) follows immediately.

Taking the mth derivative over λ of (3.39), $m = 0, 1, \ldots,$ and using (3.38) and Lemma 2, we obtain for $|\phi| < (n_0 + n + 1)^2$

$$\begin{aligned}\left|\frac{\partial^m\tilde{v}_{n+1}(\phi)}{\partial\lambda^m}\right| &\leqslant \left[L^{d/2}(1-2\epsilon)\right]^{-4}\frac{1}{1-L^{-d/2}/(1-2\epsilon)}\cdot 6L^d\eta_n C_n'^m(m!)^2\\ &\leqslant \eta_{n+1}C_n'^m(m!)^2\end{aligned}\tag{3.42}$$

which implies (3.7) for $n+1$.

Similarly, denoting

$$\delta\epsilon_{n+1} = -L^d\frac{\partial^2}{\partial\phi^2}\restriction_{\phi=0}\left[\log\int\prod_i \tilde{g}_n(z_i)\, d\mu_{\epsilon_n^{-1}}(z)\right] \tag{3.43}$$

and using the Cauchy formula, we obtain

$$\begin{aligned}\left|\frac{d^m}{d\lambda^m}\delta\epsilon_{n+1}\right| &\leqslant 12L^d\eta_n\left(L^{d/2}(1-2\epsilon)(n_0+n+1)^2\right)^{-2}C_n^{\prime m}(m!)^2\\ &\leqslant \eta_n C_n^{\prime m}(m!)^2\end{aligned}\tag{3.44}$$

From (3.43), (3.16), and (3.15), we infer that

$$\epsilon_{n+1}=\epsilon_n+\delta\epsilon_{n+1}\tag{3.45}$$

or

$$\epsilon_{n+1}^{-1}=\epsilon_n^{-1}+\epsilon_n^{-1}\left[\left(1+\epsilon_n^{-1}\delta\epsilon_{n+1}\right)^{-1}-1\right]\tag{3.46}$$

Now, for $m=0,1,2,\ldots$

$$\begin{aligned}&\left|\frac{d^m}{d\lambda^m}\epsilon_n^{-1}\delta\epsilon_{n+1}\right|\\ &\quad\leqslant\left|\frac{d^m}{d\lambda^m}\left(\epsilon_n^{-1}-1\right)\delta\epsilon_{n+1}\right|+\left|\frac{d^m}{d\lambda^m}\delta\epsilon_{n+1}\right|\\ &\quad\leqslant\sum_{p=0}^{m}\binom{m}{p}\left|\frac{d^p}{d\lambda^p}\left(\epsilon_n^{-1}-1\right)\frac{d^{m-p}}{d\lambda^{m-p}}\delta\epsilon_{n+1}\right|+\eta_n C_n^{\prime m}(m!)^2\\ &\quad\leqslant E_n\eta_n C_n^{\prime m}\sum_{p=0}^{m}m!\,p!\,(m-p)!+\eta_n C_n^{\prime m}(m!)^2\leqslant(m+2)\eta_n C_n^{\prime m}(m!)^2\end{aligned}\tag{3.47}$$

But

$$m+2\leqslant\frac{a^{m+2}}{e\log a}\tag{3.48}$$

for $a>1$. Take $a=1+(n_0+n)^{-2}$. Thus

$$\left|\frac{d^m}{d\lambda^m}\epsilon_n^{-1}\delta\epsilon_{n+1}\right|\leqslant\eta_n^{2/3}C_n^{\prime m}(m!)^2\tag{3.49}$$

Applying again Lemma 1, we obtain

$$\left|\frac{\partial^m}{\partial\lambda^m}\left[\left(1+\epsilon_n^{-1}\delta\epsilon_{n+1}\right)^{-1}-1\right]\right|\leqslant 3\eta_n^{2/3}C_n^{\prime m}(m!)^2\tag{3.50}$$

Repetition of the argument of (3.48) gives now

$$\left|\frac{\partial^m}{\partial\lambda^m}\epsilon_n^{-1}\left[\left(1+\epsilon_n^{-1}\delta\epsilon_{n+1}\right)^{-1}-1\right]\right|\leqslant E_n(n_0+n+1)^{-3/2}C_n^{\prime m}(m!)^2\tag{3.51}$$

(3.52) and (3.6) of (B_n) give (3.6) for $n+1$ and complete the inductive proof of (B_{n+1}).

Large Fields

We are left with the proof of (A_{n+1}), given (A_n), (B_n). First, notice that for $|\phi|<(n_0+n+1)^2$, $|\tilde{v}_{n+1}(\phi)/\phi^4|\leqslant\eta_{n+1}(n_0+n+1)^{-8}$ by the maximum principle and (3.43), so that

$$|\tilde{v}_{n+1}(\phi)| \leqslant \eta_{n+1}\left[\frac{|\phi|}{(n_0+n+1)^2}\right]^4 \leqslant \eta_{n+1}\left[\frac{|\phi|}{(n_0+n+1)^2}\right]^2 \leqslant \kappa_{n+1}|\phi|^2 \tag{3.52}$$

and

$$|\tilde{g}_{n+1}(\phi)| \leqslant e^{\kappa_{n+1}|\phi|^2} \tag{3.53}$$

Similarly, using Lemma 1 and (3.43), we obtain for $m = 1, 2, \ldots$

$$\left|\frac{\partial^m}{\partial\lambda^m}\tilde{g}_{n+1}(\phi)\right| \leqslant 2\eta_{n+1}(C_n')^m(m!)^2 \leqslant C_{n+1}^m(m!)^2 e^{\kappa_{n+1}|\phi|^2} \tag{3.54}$$

So we are left only with proving (3.5) for $n+1$ and $|\mathrm{Im}\,\phi| < n_0+n+1$, $|\phi| \geqslant n_0+n+1$. We shall estimate $|\partial^m h(\phi)/\partial\lambda^m|$ first. Notice that the integrand on the right-hand side of (3.18) is bounded as in (3.32), but now

$$\begin{aligned}\exp(\kappa_n|\phi|^2 + 2L^d\kappa_n|\zeta|^2) &\leqslant \exp\left\{\kappa_n\left[1+\frac{1}{2}(n_0+n+1)^{-3/2}\right]|\phi|^2\right\}\\ &\quad\times\exp\left[\tfrac{1}{2}\kappa_n(n_0+n+1)^{5/2} + 2L^d\kappa_n(n_0+n)^2\right]\\ &\leqslant \exp(\kappa_{n+1}'|\phi|^2)\exp\left[-\tfrac{1}{4}\kappa_n(n_0+n+1)^{5/2}\right]\end{aligned} \tag{3.55}$$

Estimating the sums of (3.18) as before using now the extra strength coming from the last factor on the right of (3.56), we obtain

$$\left|\frac{\partial^m h(\phi)}{\partial\lambda^m}\right| \leqslant C_n'^m(m!)^2\exp(\kappa_{n+1}'|\phi|^2)\exp\left[-\frac{1}{8}\kappa_n(n_0+n+1)^{5/2}\right] \tag{3.56}$$

for $m = 0, 1, \ldots$ and $|\phi| \geqslant (n_0+n+1)^2$, $|\mathrm{Im}\,\phi| < n_0+n+1$. But

$$\tilde{g}_{n+1}(\phi) = h(\phi)\exp(\tfrac{1}{2}L^{-d}\delta\epsilon_{n+1}\phi^2)h(0)^{-1} \tag{3.57}$$

Let us estimate using (3.45)

$$\begin{aligned}&\left|\frac{\partial^m}{\partial\lambda^m}\exp\left(\frac{1}{2}L^{-d}\delta\epsilon_{n+1}\phi^2\right)\right|\\ &\quad\leqslant \sum_{\substack{\{I_\alpha\}_{\alpha=1}^l\\ \bigcup_\alpha I_\alpha=\{1,\ldots,m\}}}\prod_{\alpha=1}^l\left|\frac{\partial^{|I_\alpha|}}{\partial\lambda^{|I_\alpha|}}\left(\frac{1}{2}L^{-d}\delta\epsilon_{n+1}\phi^2\right)\right|\\ &\qquad\times\left|\exp\left(\frac{1}{2}L^{-d}\delta\epsilon_{n+1}\phi^2\right)\right|\\ &\quad\leqslant \sum_{\substack{(m_\alpha)_{\alpha=1}^l,\, m_\alpha>0\\ \sum m_\alpha=m}}\frac{m!}{\prod_\alpha m_\alpha!\,l!}\left[\prod_\alpha\frac{1}{2}L^{-d}\eta_n C_n'^{m_\alpha}(m_\alpha!)^2|\phi|^2\right]\exp\left(\frac{1}{2}L^{-d}\eta_n|\phi|^2\right)\end{aligned} \tag{3.58}$$

Using the inequality $(\frac{1}{2}L^{-d}\eta_n^{1/2}|\phi|^2)^l \leqslant l!\exp(\frac{1}{2}L^{-d}\eta_n^{1/2}|\phi|^2)$, we obtain ($m = 1,2,\ldots$)

$$\begin{aligned}\left|\frac{\partial^m}{\partial\lambda^m}\exp\left(\frac{1}{2}L^{-d}\delta\epsilon_{n+1}\phi^2\right)\right| &\leqslant C_n^{\prime m}(m!)^2\exp\left(\eta_n^{1/2}|\phi|^2\right)\sum_{l=1}^{m}\binom{m-1}{l-1}\eta_n^{l/2}\\ &\leqslant \eta_n^{1/2}C_n^{\prime m}(m!)^2\left(1-\eta_n^{1/2}\right)^{m-1}\exp\left(\eta_n^{1/2}|\phi|^2\right)\\ &\leqslant C_n^{\prime m}(m!)^2\exp\left(\eta_n^{1/2}|\phi|^2\right) \qquad (3.59)\end{aligned}$$

which holds also for $m = 0$. Finally, using (3.37) and Lemma 1, we may write

$$\left|\frac{\partial^m}{\partial\lambda^m}h(0)^{-1}\right| \leqslant 2C_n^{\prime m}(m!)^2 \qquad (3.60)$$

for $m = 0, 1, \ldots$. Equations (3.57), (3.56), (3.59), and (3.60), give

$$\begin{aligned}&\left|\frac{\partial^m}{\partial\lambda^m}\tilde{g}_{n+1}(\phi)\right|\\ &\quad\leqslant 2\exp\left[-\frac{1}{8}\kappa_n(n_0+n+1)^{5/2}\right]C_n^{\prime m}\exp\left[\left(\kappa_{n+1}^{\prime}+\eta_n^{1/2}\right)|\phi|^2\right]\\ &\quad\times\sum_{\substack{I_1,I_2,I_3\\ \cup I_i=\{1,\ldots,m\}}}(|I_1|!)^2(|I_2|!)^2(|I_3|!)^2\\ &\quad\leqslant 2\binom{m+2}{2}\exp\left[-\frac{1}{8}\kappa_n(n_0+n+1)^{5/2}\right]C_n^{\prime m}(m!)^2\exp\left(\kappa_{n+1}|\phi|^2\right) \qquad (3.61)\end{aligned}$$

Absorbing $\binom{m+2}{2}$ into increase of $C_n^{\prime}$ with the use of (3.48), we obtain

$$\left|\frac{\partial^m}{\partial\lambda^m}\tilde{g}_{n+1}(\phi)\right| \leqslant C_{n+1}^{m}(m!)^2\exp\left(\kappa_{n+1}|\phi|^2\right) \qquad (3.62)$$

for $|\mathrm{Im}\,\phi| < (n_0+n+1)^2$, $|\phi| \geqslant (n_0+n+1)^2$ which was the missing part of (3.5) of (A_{n+1}). This completes the proof of Proposition 1. ■

4. PROOF OF THE MAIN RESULT

First we have to show that $v(\phi) = \lambda\phi^4$ satisfies (A_0), (B_0) with κ_0 small, n_0 and C_0 sufficiently large and R sufficiently small. In fact,

$$\begin{aligned}\left|\frac{\partial^m}{\partial\lambda^m}e^{-\lambda\phi^4}\right| &\leqslant \left|\phi^{4m}e^{-\lambda\phi^4}\right| \leqslant \left(\frac{1}{4}C_0\right)^m\left|2C_0^{-1/2}\phi^2\right|^{2m}\left|e^{-\lambda\phi^4}\right|\\ &\leqslant (2m)!\left(\frac{1}{4}C_0\right)^m e^{2C_0^{-1/2}|\phi|^2}\left|e^{-\lambda\phi^4}\right| \qquad (4.1)\end{aligned}$$

Setting $\lambda = \lambda_i + i\lambda_2$, $\phi = \phi_1 + i\phi_2$, we obtain

$$\begin{aligned}-\mathrm{Re}\left[(\lambda_1+i\lambda_2)(\phi_1+i\phi_2)^4\right] &= -\lambda_1\phi_1^4 + 4\lambda_2\phi_1^3\phi_2\\ &\quad+6\lambda_1\phi_1^2\phi_2^2 - 4\lambda_2\phi_1\phi_2^3 - \lambda_1\phi_2^4 \qquad (4.2)\end{aligned}$$

For $\mathrm{Re}(1/\lambda) > 1/R$, $\lambda_1 > 0$, $\lambda_1/(\lambda_1^2+\lambda_2^2) > 1/R$, taking also $|\phi_2| < n_0^2$, we see that

$$\begin{aligned} 6\lambda_1\phi_1^2\phi_2^2 - 4\lambda_2\phi_1\phi_2^3 - \lambda_1\phi_2^4 &\leqslant 6\lambda_1(\phi_1^2+\phi_2^2)\phi_2^2 + 4|\lambda_2|\,|\phi_1\phi_2|\phi_2^2 \\ &\leqslant 6Rn_0^4(\phi_1^2+\phi_2^2) + 2Rn_0^4|\phi_1\phi_2| \\ &\leqslant 7Rn_0^4(\phi_1^2+\phi_2^2) = 7Rn_0^4|\phi|^2 \end{aligned} \tag{4.3}$$

Combination of (4.1)–(4.3) gives

$$\left|\frac{\partial^m}{\partial\lambda^m} e^{-\lambda\phi^4}\right| \leqslant C_0^m(m!)^2 e^{(2C_0^{-1/2}+7Rn_0^4)|\phi|^2} \tag{4.4}$$

We shall take C_0 big enough so that $C_0^{-1/2} \leqslant \frac{1}{4}\kappa_0$ and then $R \leqslant \frac{1}{14}\kappa_0 n_0^{-4}$ to obtain

$$\left|\frac{\partial^m}{\partial\lambda^m} e^{-\lambda\phi^4}\right| \leqslant C_0^m(m!)^2 e^{\kappa_0|\phi|^2} \tag{4.5}$$

Taking $C_0 \geqslant n_0^8\delta^{-n_0}$ and $R \leqslant C_0^{-1}n_0^{-8}\delta^{n_0}$, we also obtain

$$\left|\frac{\partial^m}{\partial\lambda^m}\lambda\phi^4\right| \leqslant \delta^{n_0}C_0^m(m!)^2 \tag{4.6}$$

for $|\phi| < n_0^2$.

This establishes (A_0), (B_0) for $v(\phi) = \lambda\phi^4$. So also (A_n), (B_n) hold for all n. Since by Vitali's theorem $\epsilon_n \to \epsilon_\infty$ uniformly on C_R, (3.14) gives

$$\left|\frac{d^m}{d\lambda^m}(\epsilon_\infty - 1)\right| \leqslant 2E_\infty(2C_\infty)^m(m!)^2 \tag{4.7}$$

As far as the free energy density is concerned, notice that

$$\begin{aligned} &\log\int \exp\left[-L^d v_n(z) - \frac{1}{2}z^2\right]\frac{dz}{(2\pi)^{1/2}} \\ &\quad = \log\int \exp\left[-L^d v_n(z)\right] d\mu_{\epsilon_n^{-1}}(z) + \frac{1}{2}\log\epsilon_n^{-1} \end{aligned} \tag{4.8}$$

The λ derivatives of the first term on the right-hand side of (4.8) are bounded by (3.38). For the second term, we get using Lemma 1 and (3.6):

$$\left|\frac{\partial^m}{\partial\lambda^m}\log\epsilon_n^{-1}\right| \leqslant 2E_\infty(2C_\infty)^m(m!)^2 \tag{4.9}$$

By (2.20), f_∞ exists, is analytic on C_R, and satisfies

$$\left|\frac{\partial^m}{\partial\lambda^m} f_\infty\right| \leqslant 2E_\infty(2C_\infty)^m(m!)^2 \tag{4.10}$$

It is easy to see that from the uniform bounds (4.7) and (4.10) in C_R it follows that ϵ_∞ and f_∞ can be continued to $\overline{C}_R$ to a C^∞ function. But from our renormalization group analysis, we infer that ϵ_∞ and f_∞ are continuous (hence C^∞) at $\lambda = +0$. Now (2.22) and (2.23) easily follow for ϵ_∞ and f_∞

from the Taylor expansion with remainder and (4.7) and (4.10) and yield by N–S theorem our main result.

ACKNOWLEDGMENTS

One of the authors (K.G.) would like to thank the Institute of Physics and another author (A.K.) the Institute of Mathematics of the University of Rome for a kind invitation.

REFERENCES

1. F. J. Dyson, Divergence of perturbation theory in quantum electrodynamics, *Phys. Rev.* **85**:631–632 (1952).
2. A. Jaffe, Divergence of perturbation theory for bosons, *Commun. Math. Phys.* **1**:127–149 (1965).
3. H. G. Hardy, *Divergent series* (Oxford U.P., London, 1949).
4. J.-P. Eckmann, J. Magnen, and R. Sénéor, Decay properties and Borel summability for the Schwinger functions in $P(\phi)_2$ theories, *Commun. Math. Phys.* **39**:251–271 (1975).
5. J. Magnen and R. Sénéor, Phase space cell expansion and Borel summability for the Euclidean φ_3^4 theory, *Commun. Math. Phys.* **56**:237–276 (1977).
6. K. G. Wilson and J. Kogut, The renormalization group and the ϵ expansion, *Phys. Rep.* **12**:75–199 (1974).
7. G. Benfatto, M. Cassandro, G. Gallavotti, F. Nicolò, E. Olivieri, E. Presutte, and E. Scacciatelli, Ultraviolet stability in Euclidean field theories, *Commun. Math. Phys.* **71**:95–130 (1980).
8. J. Bricmont, J.-R. Fontaine, J. L. Lebowitz, and T. Spencer, Lattice systems with a continuous symmetry. I. Perturbation theory for unbounded spins, *Commun. Math. Phys.* **78**:281–302 (1980), and II. Decay of correlations, *Commun. Math. Phys.* **78**:363–371 (1981).
9. K. Gawędzki and A. Kupiainen, A rigorous block spin approach to massless lattice theories, *Commun. Math. Phys.* **77**:31–64 (1980).
10. J. Magnen and R. Sénéor, The infrared behaviour of $(\nabla\phi)_3^4$, to appear in *Ann. Phys. (N.Y.)*
11. K. Gawędzki and A. Kupiainen, Block spin renormalization group for dipole gas and $(\nabla\phi)^4$, *Ann. Phys.* **147**:198–243 (1983).
12. J.-R. Fontaine, Bounds on the decay of correlations for $\lambda(\nabla\phi)^4$ models, *Commun. Math. Phys.* **87**:385–394 (1982).
13. K. Gawędzki and A. Kupiainen, Lattice dipole gas and $(\nabla\phi)^4$ models at long distances: decay of correlations and scaling limit, to appear in *Commun. Math. Phys.*
14. P. M. Bleher, O fazavyh perehodah vtorogo roda v asimptoticheskih herarhicheskih modeljah Dysona, *Usp. Mat. Nauk* **32**:243–244 (1977).
15. K. Gawędzki and A. Kupiainen, Triviality of ϕ_4^4 and all that in a hierarchical model approximation, *J. Stat. Phys.* **29**:683–698 (1982).
16. F. David, Non-perturbative effects and infrared renormalons within the $1/N$ expansion of the $O(N)$ non-linear sigma model, *Nucl. Phys. B* **209**:433–460 (1982).
17. F. J. Dyson, Existence of a phase transition in a one-dimensional Ising ferromagnet, *Commun. Math. Phys.* **12**:91–107 (1969).
18. A. D. Sokal, An improvement of Watson's theorem on Borel summability, *J. Math. Phys.* **21**:261–263 (1980).